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Weighted Polynomial Approximation of Entire Functions, I

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Necessary and sufficient conditions are given for a function f defined almost everywhere on the whole real line to have an extension to the complex plane as an entire function of order 1 and finite type. These conditions are in terms of the degree of approximation of f by polynomials in weighted L^p norms. In the case of the Hermite weight, an explicit formula for the type of the extension is given.

1. INTRODUCTION

A theorem of S. N. Bernstein characterizes the type and order of an entire function in terms of the constructive properties of its restriction to $[-1, 1]$. Suppose f is a real-valued continuous function on $[-1, 1]$. Let

$$E_n(f) = \min \max_{-1 \leqslant x \leqslant 1} |f(x) - P(x)|, \tag{1}$$

where the min is taken over all algebraic polynomials P of degree at most n. The theorem of Bernstein can then be stated as follows:

THEOREM 1 [1]. *Let $f \in C[-1, 1]$. If, for some $\Lambda > 0$,*

$$\limsup_{n \to \infty} \{n!^{1/\Lambda} E_n(f)\}^{1/n} < \infty, \tag{2}$$

then f has an extension to the complex plane as an entire function $f(z)$ of order Λ and of finite type, i.e.,

$$\limsup_{R \to \infty} \frac{\log \max_{|z| \leqslant R} |f(z)|}{R^\Lambda} < \infty. \tag{3}$$

Conversely, if f is the restriction to $[-1, 1]$ of an entire function of positive order Λ and finite type, then (2) holds.

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In this paper, we seek to give necessary and sufficient conditions for a function f defined almost everywhere on the whole real line to have a entire extension of order 1 and finite type. These conditions will be given in terms of the degree of approximation of f by polynomials in weighted L^p norms.

2. Main Results

We consider weights of the form $w_Q(x) = \exp(-Q(x))$.

DEFINITION 1. A weight function $w_Q(x)$ is said to be in the class VSR (very strongly regular) if $Q(x)$ satisfies the following conditions:

$VSR1$: $Q: \mathbb{R} \to \mathbb{R}$ is an even function in $C^2(0, \infty)$.

$VSR2$: Q'' is positive and nondecreasing on $(0, \infty)$.

$VSR3$: $1 \leqslant c_1 \leqslant x \dfrac{Q''(x)}{Q'(x)} \leqslant c_2, \qquad x \in (0, \infty)$.

Remarks on the class VSR. 1. Weights of the form $\exp(-c\,|x|^\alpha)$, where $c \in (0, \infty)$, are in VSR if $\alpha \geqslant 2$. 2. The conditions, especially $VSR2$ and $VSR3$, could have been replaced by weaker conditions in terms of the growth of some Christoffel functions for the weight w_Q^2. However, these are somewhat complicated to state. We shall give quotations at appropriate places where these weaker conditions are in fact the ones which are used.

For $w_Q \in VSR$, let q_n be the least positive solution of the equation $xQ'(x) = n$:

$$q_n Q'(q_n) = n \tag{4}$$

For a Lebesgue measurable g on $\mathbb{R}$, put

$$\|g\|_p = \left\{ \int_{\mathbb{R}} |g(x)|^p \, dx \right\}^{1/p};$$

if $1 \leqslant p < \infty$,

$$\|g\|_\infty = \operatorname*{vrai\,sup}_{x \in \mathbb{R}} |g(x)|.$$

If $w_Q f \in L^p(\mathbb{R})$ and n is a nonnegative integer, put

$$\varepsilon_n(p, w_Q, f) = \inf \|w_Q(f - P)\|_p, \tag{5}$$

where the inf is taken over all polynomials P of degree at most $n-1$. We denote the class of all polynomials of degree at most n by π_n. Define

$$\underline{|q_n} = q_n q_{n-1} \cdots q_1, \qquad n! = n(n-1) \cdots 1; \qquad n \geqslant 1.$$

We can now formulate our theorem as follows:

THEOREM 2. *Suppose* $w_Q \in VSR$. *Let* $p \geqslant 1$ *and* $w_Q f \in L^p(\mathbb{R})$.

(a) f *has an extension to an entire function of order one and finite type, i.e.,*

$$\sigma(f) = \limsup_{R \to \infty} \frac{\log \max_{|z|=R} |f(z)|}{R} < \infty \tag{6}$$

if

$$\rho(p, f) = \limsup_{n \to \infty} \left\{ \frac{n!}{\lfloor q_n} \varepsilon_n(p, w_Q, f) \right\}^{1/n} < \infty. \tag{7}$$

Conversely, if f *is an entire function of order one and finite type, then for its restriction to the real line (to be denoted by* f *again), we have* $w_Q f \in L^p(\mathbb{R})$ *for each* $p \geqslant 1$ *and*

$$\rho(p, f) < \infty, \qquad p \geqslant 1.$$

(b) *If* f *is an entire function of order one and finite type* $\sigma(f)$, *then there exist positive constants* c_3 *and* c_4 *depending upon* Q *but not on* f *or* p *such that*

$$c_3 \sigma(f) \leqslant \rho(p, f) \leqslant c_4 \sigma(f), \qquad p \geqslant 1. \tag{8}$$

(c) *If* $Q(x) = x^2/2$, *then* $\rho(p, f) = \sigma(f)/2^{1/2}$, $p \geqslant 1$. (9)

3. PROOFS

We shall prove the theorem first for $p = 2$ and then extend it to other values of p. It is convenient to do so since the Parseval's formula gives a precise expression for the degree of approximation in the $L^2(\mathbb{R})$ norm.

Let $\{p_k(w_Q^2, x) = p_k(x)\}_{k=0}^{\infty}$ be the family of orthonormal polynomials with respect to the weight $w_Q^2(x)$. For $w_Q f \in L^p(\mathbb{R})$, $p \geqslant 1$, we have the Fourier orthonormal expansion

$$f(x) \sim \sum b_k p_k(x), \tag{10}$$

where

$$b_k = \int_{\mathbb{R}} f(t)\, p_k(t)\, w_Q^2(t)\, dt. \tag{11}$$

We then have, for $w_Q f \in L^2(\mathbb{R})$,

$$\varepsilon_n(2, w_Q, f) = \left(\sum_{k=n}^{\infty} b_k^2 \right)^{1/2}. \tag{12}$$

Before we proceed, let us recollect a proposition.

PROPOSITION 1. (a) *There exists a constant c_5 such that for every $P \in \pi_k$,*

$$\| w_Q P' \|_2 \leqslant c_5 (k/q_k) \| w_Q P \|_2. \tag{13}$$

For $Q(x) = x^2/2$, in which case the p_k's are the orthonormal Hermite polynomials, one shows by straightforward computation that c_5 can be chosen to be $\sqrt{2}$ (and no less).

(b) *There exist constants c_6, c_7 such that for each polynomials $P \in \pi_n$,*

$$c_6 \left(\frac{1}{q_n} \right)^{1/p - 1/r} \| w_Q P \|_p \leqslant \| w_Q P \|_r \leqslant c_7 \left(\frac{n}{q_n} \right)^{1/p - 1/r} \| w_Q P \|_p, \tag{14}$$

where $1 \leqslant p < r \leqslant \infty$ and c_6, $c_7 > 0$ depend only on Q, p and r.

Proof. Part (a) is a special case of the Markov–Berstein-type inequalities obtained by Freud [3]. Part (b) is given in our paper [6]. ∎

LEMMA 1. *Suppose $\rho(p, f)$ in (7) is finite for $p = 2$. Let $\mu > \rho(2, f)$. Then there exists a constant $c_8 = c_8(\mu, Q, f)$ such that for sufficiently large n,*

$$\sum_{k=n}^{\infty} |b_k| \, \| w_Q p_k^{(n)} \|_\infty \leqslant c_8 c_5^n \mu^n, \tag{15}$$

where b_k's are defined by (11).

Proof. Choose N such that $n \geqslant N$ implies

$$|b_k| \leqslant \varepsilon_n(2, w_Q, f) \leqslant \mu^n \, \underline{|q_n} / n!, \qquad k \geqslant n. \tag{16}$$

Now, $p_k^{(n)} \in \pi_{k-n}$. So, by (14), (13) and (16), we get, for $n \geqslant N$,

$$\begin{aligned} \sum_{k=n}^{\infty} |b_k| \, \| w_Q p_k^{(n)} \|_\infty &\leqslant c_7 \sum_{k=n}^{\infty} \frac{\mu^k \underline{|q_k}}{k!} \left(\frac{k-n}{q_{k-n}} \right)^{1/2} \| w_Q p_k^{(n)} \|_2 \\ &\leqslant c_7 \sum_{k=n}^{\infty} \mu^k \frac{\underline{|q_k}}{k!} \left(\frac{k-n}{q_{k-n}} \right)^{1/2} \frac{k!}{\underline{|q_k}} \frac{\underline{|q_{k-n}}}{(k-n)!} c_5^n \\ &= c_7 c_5^n \mu^n \sum_{k=0}^{\infty} \left(\frac{k}{q_k} \right)^{1/2} \frac{\underline{|q_k}}{k!} \mu^k. \end{aligned} \tag{17}$$

The series converges by ratio test (say) to a constant $c_8(\mu)/c_7$, thus proving the inequality (15). ∎

In view of Lemma 1, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=n}^{\infty} |b_k| \, |p_k^{(n)}(0)| \, |z|^n$$

converges uniformly in z on compact subsets of the complex plane C. Hence, we can interchange the order of summation to get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=n}^{\infty} b_k p_k^{(n)}(0) \, z^n = \sum_{k=0}^{\infty} b_k \sum_{n=0}^{\infty} \frac{1}{n!} \, p_k^{(n)}(0) \, z^n = \sum_{k=0}^{\infty} b_k p_k(z). \quad (18)$$

The last series thus converges uniformly on compact subsets of C to an entire function $g(z)$, say. It follows that the restriction of g to the real line is almost everywhere equal to f. Further, for g we have the power series

$$g(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{k=n}^{\infty} b_k p_k^{(n)}(0) \right] z^n. \quad (19)$$

So,

$$n! \, |a_n| \leqslant \sum_{k=n}^{\infty} |b_k| \, |p_k^{(n)}(0)| \leqslant w_Q^{-1}(0) \, c_8 c_5^n \mu^n.$$

Thus, g is of order 1 and type σ given by [2, p. 11, Formula 2.2.12]

$$\sigma = \limsup_{n \to \infty} \{n! \, |a_n|\}^{1/n} \leqslant c_5 \mu < \infty.$$

Since $\mu > \rho$ was arbitrary and $c_5 = \sqrt{2}$ if $Q(x) = x^2/2$, this completes the proof of the first half of part (a), the first inequality in (8) with $c_3 = c_5^{-1}$ and a part of (9) in the case $p = 2$.

Let us now turn to the remaining parts of the theorem, in the case $p = 2$. Let f be the restriction to the real line of an entire function of order one and finite type $\sigma < \tau < \infty$. Since $w_Q \in VSR$, this implies $w_Q f \in L^2(\mathbb{R})$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n. \quad (20)$$

We have [2, p. 11. Formula 2.2.12]

$$\sigma = \limsup_{n \to \infty} \{n! \, |a_n|\}^{1/n}. \quad (21)$$

PROPOSITION 2. (a) *There exists a constant* c_9 *such that*

$$\varepsilon_k(2, w_Q, x^n) \leqslant c_9 \frac{q_k}{k} \varepsilon_k(2, w_Q, nx^{n-1}) \tag{22}$$

for each n and for each $k \leqslant n$.

(b) *If* $Q(x) = x^2/2$, *we may take* $c_9 = (\sqrt{2})^{-1}$.

Proof. Part (a) is a special case of the Jackson–Favard–type inequalities proved by Freud [4]. We prove part (b). Note that if $Q(x) = x^2/2$, then $q_k = \sqrt{k}$. Observe, by Rodrigues' formula [8, p. 106], that if $n - r$ is even, then

$$\begin{aligned}
&\int_{\mathbb{R}} x^n h_r(x)\, e^{-x^2}\, dx \\
&\quad = \frac{(-1)^r}{\pi^{1/4} 2^{r/2} r!^{1/2}} \int_{\mathbb{R}} x^n \frac{d^r}{dx^r} e^{-x^2}\, dx \\
&\quad = \frac{1}{\pi^{1/4} 2^{r/2} r!^{1/2}} \frac{n!}{(n-r)!} \int_{\mathbb{R}} x^{n-r} e^{-x^2}\, dx \quad \text{(integrating by parts } r \text{ times)} \\
&\quad = \frac{1}{\pi^{1/4} 2^{r/2} r!^{1/2}} \frac{n!}{(n-r)!} \Gamma\left(\frac{n-r+1}{2}\right).
\end{aligned} \tag{23}$$

So, denoting the Hermite weight by w_2,

$$\begin{aligned}
&\varepsilon_{2k}^2(2, w_2, x^{2n}) \\
&\quad = \frac{(2n)!^2}{\sqrt{\pi}} \sum_{r=k}^{n} \frac{1}{2^{2r}(2r)!} \frac{1}{(2n-2r)!^2} \Gamma(n + \tfrac{1}{2} - r)^2 \\
&\quad = (2n)^2 \frac{(2n-1)!^2}{\sqrt{\pi}} \sum_{r=k}^{n} \frac{1}{2(2r)} \frac{1}{2^{2r-1}(2r-1)!} \frac{\Gamma(n + \frac{1}{2} - r)^2}{(2n-2r)!^2} \\
&\quad \leqslant \frac{(2n)^2}{2(2k)} \frac{(2n-1)!^2}{\sqrt{\pi}} \sum_{r=k}^{n} \frac{1}{2^{2r-1}(2r-1)!} \frac{\Gamma(n + \frac{1}{2} - r)^2}{(2n-2r)!^2} \\
&\quad = \frac{1}{2(2k)} \varepsilon_{2k-1}^2(2, w_2, 2nx^{2n-1}).
\end{aligned} \tag{24}$$

Similarly,

$$\varepsilon_{2k-1}^2(2, w_2, x^{2n-1}) \leqslant \frac{1}{2(2k-1)} \varepsilon_{2k-2}^2(2, w_2, (2n-1)\, x^{2n-2}). \tag{24'}$$

Finally, note that if $n-k$ is odd, then

$$\varepsilon_k(2, w_2, x^n) = \varepsilon_{k+1}(2, w_2, x^n)$$

to complete the proof. ∎

PROPOSITION 3. *There exists constants c_{11} and c_{10} depending upon Q alone such that, for each nonnegative integer r,*

$$\| w_Q x^r \|_2 \leqslant c_{11} c_{10}^r \underline{|q_r}. \tag{25}$$

Proof. Let $x_{1n} > x_{2n} > \cdots > x_{nn}$ be the zeroes of $p_n(w_Q^2, x)$ and λ_{kn} the corresponding Cote's numbers. We have, by the quadrature formula,

$$\begin{aligned} \int_{\mathbb{R}} x^{2r} w_Q^2(x)\, dx &= \sum_{k=0}^{r+1} \lambda_{k,r+1} x_{k,r+1}^{2r} \\ &\leqslant x_{1,r+1}^2 \sum_{k=0}^{r+1} \lambda_{k,r+1} x_{k,r+1}^{2r-2} = x_{1,r+1}^2 \int_{\mathbb{R}} x^{2r-2} w_Q^2(x)\, dx. \end{aligned} \tag{26}$$

The result now follows by induction of we apply the inequality [5]

$$x_{1,n} \leqslant c'_{10} q_n \leqslant c''_{10} q_{n-1}. \quad ∎ \tag{27}$$

Now, we can complete the proof. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be of type $\sigma < \infty$, $\sigma < \tau < \infty$. By (21), choose N so large that $n \geqslant N$ implies

$$|a_n| \leqslant \tau^n / n!.$$

Then

$$\sum_{n=N}^{\infty} |a_n| \, \| w_Q x^n \|_2 \leqslant c_{11} \sum_{n=N}^{\infty} \frac{\tau^n}{n!} c_{10}^n \underline{|q_n} < \infty. \tag{28}$$

(The last inequality is a consequence of, say, the ratio test.) So, if $k \geqslant N$, we get, from (22) and (25),

$$\begin{aligned} \varepsilon_k(2, w_Q, f) &\leqslant \sum_{n=k}^{\infty} |a_n| \, \varepsilon_k(2, w_Q, x^n) \\ &\leqslant \sum_{n=k}^{\infty} \frac{\tau^n}{n!} c_9^k \frac{\underline{|q_k}}{k!} \varepsilon_0(2, w_Q, x^{n-k}) \frac{n!}{(n-k)!} \\ &\leqslant c_{11} c_9^k \tau^k \frac{\underline{|q_k}}{k!} \sum_{n=k}^{\infty} \frac{\tau^{n-k}}{(n-k)!} c_{10}^{n-k} \underline{|q_{n-k}}. \end{aligned}$$

Thus,

$$\varepsilon_k(2, w_Q, f) \leqslant c_{11} c_9^k \tau^k \frac{\underline{|q_k}}{k!} \sum_{n=0}^{\infty} \frac{\tau^n c_{10}^n}{n!} \underline{|q_n}$$
$$= c_{12} c_9^k \tau^k \frac{\underline{|q_k}}{k!}, \tag{29}$$

where $c_{12} = c_{12}(\tau, Q)$ is a constant. (We have seen earlier that the last series converges.) Then,

$$\limsup_{k\to\infty} \left\{ \frac{k!}{\underline{|q_k}} \varepsilon_k(2, w_Q, f) \right\}^{1/k} \leqslant c_9 \tau.$$

Since $\tau > \sigma$ was arbitrary, we get

$$\rho(2, f) \leqslant c_9 \sigma(f). \tag{30}$$

This proves the theorem in the case $p = 2$. Observe that $c_3 = c_5^{-1}$ and $c_4 = c_9$, where c_5 and c_9 were defined in Propositions 1 and 2, respectively. Further, if $w_Q = w_2$, the Hermite weight, then $c_5 = \sqrt{2}$ and $c_9 = 1/\sqrt{2}$, which gives part (c) of the theorem in the case $p = 2$.

We now proceed to extend the theorem from the case $p = 2$ to the case of arbitrary $p \geqslant 1$. In fact, we shall show that the quantity $\rho(p, f)$ of (7) is really independent of p as long as $p \geqslant 1$; i.e., if $1 \leqslant p, r \leqslant \infty$, then

$$\rho(p, f) = \rho(r, f). \tag{31}$$

(cf. [7])

LEMMA 2. (a) *There exists a constant* c_{13} *such that for all* $n \in \mathbb{N}$,

$$q_n \leqslant c_{13} \frac{n}{q_n}. \tag{32}$$

(b) *If* $\alpha > 0$ *and* $\tau \in \mathbb{R}$, *then there exists a* $K = K(\alpha, \tau, Q)$ *such that* $k \geqslant K$ *implies*

$$\sum_{n=k}^{\infty} \left(\frac{\underline{|q_n}}{n!} \right)^{\alpha} \tau^n \leqslant 2 \left(\frac{\underline{|q_k}}{k!} \right)^{\alpha} \tau^k. \tag{33}$$

Proof. (a) By conditions $VSR3$ and $VSR2$ on w_Q,

$$\frac{q_n^2}{n} = \frac{q_n}{Q'(q_n)} \leqslant \frac{c_2}{Q''(q_n)} \leqslant c_{13}.$$

(b)

$$\left(\frac{\underline{|q_k}}{k!}\right)^\alpha \tau^k + \left(\frac{\underline{|q_{k+1}}}{(k+1)!}\right)^\alpha \tau^{k+1} + \cdots$$

$$= \left(\frac{\underline{|q_k}}{k!}\right)^\alpha \tau^k \left\{1 + \frac{\tau}{\left[\frac{(k+1)}{q_{k+1}}\right]^\alpha} + \frac{\tau^2}{\left[\frac{(k+2)}{q_{k+2}}\cdot\frac{(k+1)}{q_{k+1}}\right]^\alpha} + \cdots\right\}$$

$$\leqslant \left(\frac{\underline{|q_k}}{k!}\right)^\alpha \tau^k \left\{1 + \frac{\tau}{\left[\frac{(k+1)}{q_{k+1}}\right]^\alpha} + \left(\frac{\tau}{\left[\frac{(k+1)}{q_{k+1}}\right]^\alpha}\right)^2 + \cdots\right\}$$

$$\leqslant 2\left(\frac{\underline{|q_k}}{k!}\right)^\alpha \tau^k$$

if k is large enough. ∎

Now suppose

$$\rho(p,f) = \limsup_{n\to\infty}\left\{\frac{n!}{\underline{|q_n}}\,\varepsilon_n(p, w_Q, f)\right\}^{1/n} < \infty$$

for some $p \geqslant 1$.

Let $\rho < \mu < \infty$ and $\tau_n \in \pi_n$ be chosen such that

$$\|w_Q(f-\tau_n)\|_p \leqslant 2\varepsilon_n(p, w_Q, f). \tag{34}$$

Writing $P_n = \tau_{n+1} - \tau_n$, we get

$$f = \sum_{n=0}^{\infty} P_n + \tau_0 \tag{35}$$

in the sense that

$$\left\| w_Q\left(f - \tau_0 - \sum_{k=0}^{N} P_k\right)\right\|_p \to 0 \text{ as } N \to \infty. \tag{35$'$}$$

Also,

$$\|w_Q P_n\| \leqslant 4\varepsilon_n(p, w_Q, f). \tag{36}$$

Now, if $r \geqslant 1$, we have, by (14) (Proposition 1, part b),

$$\|w_Q P_n\|_r \leqslant c_{14} q_n^{1/p-1/r}\|w_Q P_n\|_p \qquad \text{if} \quad r \leqslant p,$$

$$\|w_Q P_n\|_r \leqslant c_7\left(\frac{n}{q_n}\right)^{1/p-1/r}\|w_Q P_n\|_p \qquad \text{if} \quad r \geqslant p.$$

But, in any case, by inequality (32) (Lemma 2, part a),

$$\|w_Q P_n\|_r \leqslant c_{15} \left(\frac{n}{q_n}\right)^{1/p-1/r} \|w_Q P_n\|_p$$

$$\leqslant c_{16} \left(\frac{n}{q_n}\right) \|w_Q P_n\|_p, \qquad p, r \geqslant 1. \tag{37}$$

(The last inequality holds since

$$1/p - 1/r - 1 = (1/p - 1) - 1/r \leqslant 0 \text{ and } n/q_n \to \infty \text{ as } n \to \infty.)$$

Then for large enough k,

$$\sum_{n=k}^{n} \|w_Q P_r\|_r \leqslant c_{17} \sum_{n=k}^{\infty} \frac{n}{q_n} \cdot \frac{\underline{|q_n}}{n!}$$

$$= c_{17}\mu \sum_{n=k}^{\infty} \frac{\underline{|q_{n-1}}}{(n-1)!} \mu^{n-1} \leqslant c_{18}\mu^k \frac{\underline{|q_{k-1}}}{(k-1)!}$$

(By (33), Lemma 2, part b).

Since $\sum \underline{|q_{k+1}}/(k-1)!\,\mu^k$ converges (say, by the ratio test), this implies that $\sum w_Q P_n$ converges in $L^r(\mathbb{R})$ and also in the $L^p(\mathbb{R})$ norm. In the $L^p(\mathbb{R})$ norm, it converges to fw_Q. So,

$$w_Q f = \tau_0 w_Q + \sum_{n=0}^{\infty} w_Q P_n$$

in the $L^r(\mathbb{R})$ space and, for large k,

$$\varepsilon_k(r, w_Q, f) \leqslant \sum_{n=k-1}^{\infty} \|w_Q P_n\|_r \leqslant c_{18}\mu^{k-1} \frac{\underline{|q_{k-2}}}{(k-2)!}.$$

Therefore,

$$\frac{k!}{\underline{|q_k}} \varepsilon_k(r, w_Q, f) \leqslant \left[\frac{c_{18}(k+1)k}{\mu q_{k+1} q_k}\right] \mu^k = g(k)\mu^k \qquad \text{(say)}.$$

Now, from (32), $\lim_{k\to\infty} [g(k)]^{1/k} = 1$. So, since $\mu > \rho(p, f)$ was arbitrary,

$$\rho(r, f) = \limsup_{k\to\infty} \left\{\frac{k!}{\underline{|q_k}} \varepsilon_k(r, w_Q, f)\right\}^{1/k}$$

$$\leqslant \limsup_{k\to\infty} \left\{\frac{k!}{\underline{|q_k}} \varepsilon_k(p, w_Q, f)\right\}^{1/k} = \rho(p, f)$$

for all p, $r \geqslant 1$. This completes the proof of our theorem, even for the general values of p. ∎

4. Remarks

(1) It is our conjecture that the constants c_3 and c_4 appearing in (8) are equal, as they in (9). However, the proof of this seems to require a great deal of information about the asymptotic behaviour of general orthonormal polynomials with weights supported on the whole real line, which is far from being known.

(2) Using the same techniques used in this paper, we can obtain some expressions for the order and the type of an entire function in terms of $\varepsilon_n(p, w_Q, f)$, although we do not expect to have in this direction theorems as complete as part c of our theorem. We hope to return to this question soon.

(3) Part c of our theorem is interesting in that it gives an explicit expression for the type $\sigma(f)$ in terms of $\varepsilon_n(p, w_Q, f)$, even if in a special case, as contrasted to the theorem of Bernstein (Theorem 1).

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